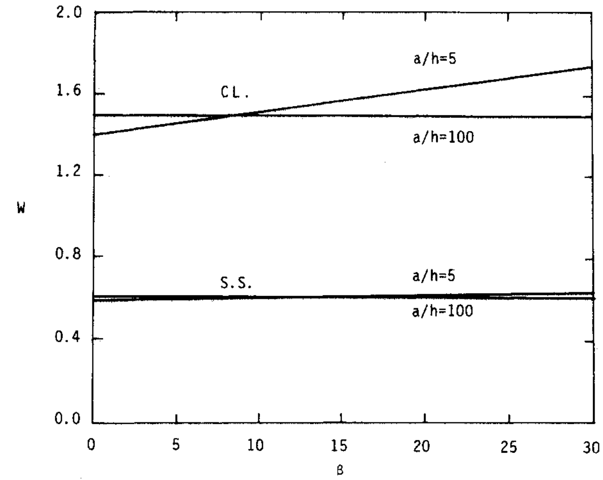

Fig. 5 Central deflection vs β for thick and thin plates; $K/K_{cr}=2.0$, $S=1.0$, $\nu=0.3$.

Conclusions

The preliminary results indicate the following.

1) By using the present method one can obtain the buckling coefficient to match Pardoen's results.

2) The thick plate buckles at a lower $-K$ than the thin plate does, and the thick plate post-buckled deflection is larger than the corresponding thin plate.

3) Deflections increase with the decreasing of the transversely isotropic coefficient S.

4) The post-buckled deflections of the plate increase when the Poisson's ratio decreases.

5) The deflection increases with increasing bending stress coefficient. The effects are significant for thick plates.

6) Thick plate effects are accentuated by increasing the boundary restraint.

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Linear System Identification via Poisson Moment Functionals

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Introduction

THE method of Poisson moment functionals (PMF) provides an effective system identification algorithm in the time domain for linear time-invariant systems. First introduced by Fairman and Shen[1,2] and Fairman,[3] the method was systematically adapted to discrete linear time-invariant and time-varying systems in a series of papers by Saha and Prasada Rao[4-8] and Sivakumar and Prasada Rao.[9] In that series of papers, they provide algorithms for the synthesis of transfer functions from time domain data, both for homogeneous initial conditions and, by an augmentation procedure, for arbitrary unknown initial conditions. More importantly, they provide an algorithm for the direct identification of the elements of the system state matrix. However, that method has been developed only for the force-free case, wherein the initial conditions are fully known.

The desirable attributes of identification via PMF are now well documented. Among them are the following.

1) Identification is done in continuous time, despite the fact that process signals are sampled in discrete time.

2) The method is somewhat naturally immune to zero-mean additive noise.

3) The PMF's of the process signals can be obtained online as the response of a Poisson filter chain.

Application of the method to the state identification of structures has been hampered by limitation of existing algorithms to the force-free case. This is nearly always too restrictive, for several reasons. First, the most accurate dynamic test methods involve one or more actuators exciting the structure; and second, the importance of substructure input matrices becomes equal to that of the state matrices when a large structure is synthesized from identified substructure parameters (see, for example, Hale and Bergman[10]).

The purpose of this Note, then, is to provide an extension of the algorithm, first proposed in Ref. 4, to the synthesis of state equations when the system to be identified is subjected to forces, internal and/or external.

Poisson Moment Functionals

The PMF transform takes a signal $f(t)$ over the interval $(0,t_0)$ and converts it into a set of real numbers

$$M_k[f(t)]_{t_0}=f_k=\int_0^{t_0} f(t)p_k(t_0-t)\,dt, \quad (k=0,1,2,\ldots) \tag{1}$$

where

$$p_k(t)=t^k\exp(-\lambda t)/k! \tag{2}$$

$$\lambda(\text{real})\geq 0, \text{ and } f_{-1}=f(t_0) \tag{3}$$

Received May 21, 1983.

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Thus, as stated in Ref. 4, M_k is a linear operator involving convolution of the process signal with the impulse response of $(k+1)$ cascaded filters, each with a transfer function $1/(s+\lambda)$.

If $f(t)$ and $df(t)/dt$ are zero for $t \leq 0$, and if $f_k^{(1)}$ denotes the kth PMF of $df(t)/dt$, then it can be shown[4] that

$$f_k^{(1)} = -\lambda f_k + f_{k-1}, \qquad (k=0,1,2,\ldots) \tag{4}$$

This property of PMF permits derivatives to be handled without direct differentiation.

PMF's About a Single Instant of Time

We wish to identify the system

$$\dot{x}(t) = [A]x(t) + [B]u(t), \qquad t>0 \tag{5}$$

from potentially noisy response data. Here, $\boldsymbol{x}(t)$ is an n-dimensional state vector normally consisting, for a structural system, of displacements and velocities; $\boldsymbol{u}(t)$ is an m-dimensional input vector consisting of forces.

Taking PMF's of the state vector $\boldsymbol{x}(t)$ and its first derivative $d\boldsymbol{x}(t)/dt$, as well as of $\boldsymbol{u}(t)$, and rearranging Eq. (5), we may write

$$[C] = [Y][\Psi][\Phi]^{-1} \tag{6}$$

where we define

$$[C] = [A \;\; B] \tag{7}$$

is an $n \times (n+m)$ matrix;

$$[Y] = [I \;\; \Lambda] \tag{8}$$

is an $n \times 2n$ matrix; $[I]$ is the $n \times n$ identify matrix and $[\Lambda] = -\lambda[I]$;

$$[\Psi] = \left[\begin{array}{cccc} x_{k-1} & x_k & --- & x_{k+(n+m)-2} \\ \hline x_k & x_{k+1} & --- & x_{k+(n+m)-1} \end{array}\right] \tag{9}$$

is a $(2n) \times (n+m)$ matrix; and

$$[\Phi] = \left[\begin{array}{cccc} x_k & x_{k+1} & --- & x_{k+(n+m)-1} \\ \hline u_k & u_{k+1} & --- & u_{k+(n+m)-1} \end{array}\right] \tag{10}$$

is an $(n+m) \times (n+m)$ matrix.

Example I

The single-degree-of-freedom system given by the second-order equation of motion

$$\ddot{x} + 100x = 10H(t) \qquad t>0$$

$$x(0) = \dot{x}(0) = 0$$

where the Heaviside function H is

$$H(t) = 1, \qquad t \geq 0$$

$$= 0, \qquad t<0$$

responds in terms of PMF with $\lambda = 1$ about $t_0 = 1$ s as

$x_{1,-1} = 1.83907153 \times 10^{-1}$ $\quad x_{2,-1} = -5.44021110 \times 10^{-1}$

$x_{1,0} = 6.97934045 \times 10^{-2}$ $\quad x_{2,0} = 1.14113749 \times 10^{-1}$

$x_{1,1} = 2.57236714 \times 10^{-2}$ $\quad x_{2,1} = 4.40697307 \times 10^{-2}$

$x_{1,2} = 7.76898929 \times 10^{-3}$ $\quad x_{2,2} = 1.79546824 \times 10^{-2}$

$u_{1,0} = 6.32120563 \times 10^{-1}$

$u_{1,1} = 2.64241117 \times 10^{-1}$

$u_{1,2} = 8.03013980 \times 10^{-2}$

where we have chosen $k=0$. Hence,

$$[\Phi] = \begin{bmatrix} 6.97934045 \times 10^{-2} & 2.57236714 \times 10^{-3} & 7.76898929 \times 10^{-3} \\ 1.14113749 \times 10^{-1} & 4.40697307 \times 10^{-2} & 1.79546824 \times 10^{-2} \\ 6.32120563 \times 10^{-1} & 2.64241117 \times 10^{-1} & 8.03013980 \times 10^{-2} \end{bmatrix}$$

$$[\Psi] = \begin{bmatrix} 1.83907152 \times 10^{-1} & 6.97934045 \times 10^{-2} & 2.57236714 \times 10^{-2} \\ -5.44021110 \times 10^{-1} & 1.14113749 \times 10^{-1} & 4.40697307 \times 10^{-2} \\ 6.97934045 \times 10^{-2} & 2.57236714 \times 10^{-2} & 7.76898929 \times 10^{-3} \\ 1.14113749 \times 10^{-1} & 4.40697307 \times 10^{-2} & 1.79546824 \times 10^{-2} \end{bmatrix}$$

$$[Y] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$[C] = \begin{bmatrix} -5.55286618 \times 10^{-7} & 9.99999773 \times 10^{-1} & 1.00464372 \times 10^{-7} \\ -1.00000003 \times 10^{2} & -8.12652842 \times 10^{-8} & 1.00000003 \times 10^{1} \end{bmatrix}$$

where the data actually belong to the system

$$[C] = \begin{bmatrix} 0 & 1 & 0 \\ -100 & 0 & 10 \end{bmatrix}$$

PMF's About Different Instants of Time

As an alternate approach, and one which becomes computationally advantageous as the dimension of the system grows large, the order of the PMF's are minimized, and data are obtained instead at the necessary number of instants of time. The new system of equations to be solved is

$$[C] = [Y][\hat{\Psi}][\hat{\Phi}]^{-1} \tag{11}$$

where $[C]$ and $[Y]$ are defined in Eqs. (7) and (8), respectively;

$$[\hat{\Psi}] = \left[\begin{array}{cccc} x_{k-1}^{t_0} & x_{k-1}^{t_1} & --- & x_{k-1}^{t_{(n+m)-1}} \\ \hline x_k^{t_0} & x_k^{t_1} & --- & x_k^{t_{(n+m)-1}} \end{array}\right] \tag{12}$$

a $(2n) \times (n+m)$ matrix; and

$$[\hat{\Phi}] = \left[\begin{array}{cccc} x_k^{t_0} & x_k^{t_1} & --- & x_k^{t_{(n+m)-1}} \\ \hline u_k^{t_0} & u_k^{t_1} & --- & u_k^{t_{(n+m)-1}} \end{array}\right] \tag{13}$$

a $(n+m) \times (n+m)$ matrix.

Example II

The previously defined second-order equation responds in terms of PMF, with $\lambda = 1$ about $t_0 = 0.5$ s, $t_1 = 1.0$ s, $t_2 = 1.5$ s, as

$x_{1,-1}^{0.5} = 7.16337814 \times 10^{-2}$ $\quad x_{2,-1}^{0.5} = -9.58924274 \times 10^{-1}$

$x_{1,-1}^{1.0} = 1.83907152 \times 10^{-1}$ $\quad x_{2,-1}^{1.0} = -5.44021110 \times 10^{-1}$

$x_{1,-1}^{1.5} = 1.75968791 \times 10^{-1}$ $\quad x_{2,-1}^{1.5} = 6.50287840 \times 10^{-1}$

$x_{1,0}^{0.5} = 4.91609053 \times 10^{-2}$ $\quad x_{2,0}^{0.5} = 2.24728763 \times 10^{-2}$

$x_{1,0}^{1.0} = 6.97934045 \times 10^{-2}$ $\quad x_{2,0}^{1.0} = 1.14113749 \times 10^{-1}$

$x^{1.5}_{1,0} = 7.22215737 \times 10^{-2}$ $\quad x^{1.5}_{2,0} = 1.03747215 \times 10^{-1}$

$u^{0.5}_{1,0} = 3.93469340 \times 10^{-1}$

$u^{1.0}_{1,0} = 6.32120563 \times 10^{-1}$

$u^{1.5}_{1,0} = 7.76869840 \times 10^{-1}$

where, again, we have chosen $k=0$. Hence,

$$[\hat{\Phi}] = \begin{bmatrix} 4.91609053 \times 10^{-2} & 6.97934045 \times 10^{-2} & 7.22215737 \times 10^{-2} \\ 2.24728763 \times 10^{-2} & 1.14113749 \times 10^{-1} & 1.03747215 \times 10^{-1} \\ 3.93469340 \times 10^{-1} & 6.32120563 \times 10^{-1} & 7.76869840 \times 10^{-1} \end{bmatrix}$$

$$[\hat{\Psi}] = \begin{bmatrix} 7.16337814 \times 10^{-2} & 1.83907152 \times 10^{-1} & 1.75968791 \times 10^{-1} \\ -9.58924274 \times 10^{-1} & -5.44021110 \times 10^{-1} & 6.50287840 \times 10^{-1} \\ 4.91609053 \times 10^{-2} & 6.97934045 \times 10^{-2} & 7.22215737 \times 10^{-2} \\ 2.24728763 \times 10^{-2} & 1.14113749 \times 10^{-1} & 1.03747215 \times 10^{-1} \end{bmatrix}$$

$$[C] = \begin{bmatrix} -1.81247486 \times 10^{-7} & 9.99999969 \times 10^{-1} & 2.38484290 \times 10^{-8} \\ -9.99999800 \times 10^{1} & -2.20344477 \times 10^{-6} & 9.99999731 \end{bmatrix}$$

where, again, for comparison, the data belong to

$$[C] = \begin{bmatrix} 0 & 1 & 0 \\ -100 & 0 & 10 \end{bmatrix}$$

Conclusion

The method of Poisson moment functionals has been extended to include synthesis of the state equations when internal and/or external forces are prescribed. The method is advantageous as process signals are exponentially weighted and integrated, thus minimizing some of the effects of zero mean additive noise.

Although the method does require numerical convolution of each process signal (input and output), or, alternately, numerical forward and inverse Laplace transformation of each process signal in order to implement the filter chain, the actual solution for the unknown coefficients involves inversion of a matrix of rank $(n+m)$. However, when more sensors are available than is necessary to determine the unknown $[C]$ matrix, the algorithm is easily modified. The matrices $[\Phi]$ and $[\Psi]$ (or $[\hat{\Phi}]$ and $[\hat{\Psi}]$) become rectangular, with more rows than columns. Solution is then obtained by the least squares formulation,

$$[C] = [Y][\Phi]^T[\Psi]([\Phi]^T[\Phi])^{-1} \tag{14}$$

Finally, PMF provides a means of identification in continuous time, eliminating the necessity of transformation from discrete to continuous time of identified parameters.

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Non-Fourier Thermal Stresses in a Circular Disk

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Introduction

OVER the past few years, research has been conducted dealing with departures from the classical Fourier conduction law. The reason for this research was to eliminate the paradox of an infinite thermal wave speed and thus provide a theory to explain the experimental data on "second sound" in materials such as liquid and solid helium at low temperatures.[1,2] In addition to low-temperature applications, non-Fourier theories may also be useful for high heat flux, short time behavior as found, for example, in laser material interaction.

Fourier conduction theory relates the heat flux vector $\boldsymbol{q}$ to the temperature gradient ∇T, by the relation

$$q = -k\nabla T \tag{1}$$

where the constant k is the thermal conductivity. Only isotropic and homogeneous media will be considered herein. Equation (1), along with the conservation of energy, gives the classical parabolic heat equation

$$\alpha \nabla^2 T = \frac{\partial T}{\partial t} \tag{2}$$

where $\alpha = k/\rho c^*$ = thermal diffusivity, ρ = mass density, and c^* = specific heat capacity. Relation (2) produces temperature solutions that correspond to an infinite speed of heat propagation.

Received March 7, 1983.

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